

FERMIONIC q -FOCK SPACE AND BRAIDED GEOMETRY

S. Majid¹

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW

+

Research Institute of Mathematical Sciences
Kyoto University, Kyoto 606, Japan

July – revised November, 1995

Abstract We write the fermionic q -Fock space representation of $U_q(\hat{sl}_n)$ as an infinite extended braided tensor product of finite-dimensional fermionic $U_q(sl_n)$ -quantum planes or exterior algebras. Using braided geometrical techniques developed for such quantum exterior algebras, we provide a new approach to the Kashiwara-Miwa-Stern action of the Heisenberg algebra on the q -fermionic Fock space, obtaining the action in detail for the lowest nontrivial case $[b_2, b_{-2}] = 2(\frac{1-q^{-4n}}{1-q^{-4}})$. Our R-matrix approach includes other Hecke R-matrices as well.

Keywords: affine quantum group – q -Fock space – fermion – braided geometry – vertex operator – R-matrix.

1 Introduction

In this note we use techniques from ‘braided geometry’ to study the q -deformed fermionic Fock space representations of the affine quantum groups $U_q(\hat{sl}_n)$ [1][2][3][4]. The properties of this q -deformed Fock space are closely connected with the theory of vertex operator algebras and q -correlation functions. In particular, using the vertex operator algebra approach it has been shown in [4] that there is an action of the Heisenberg algebra on the level 1 fermionic Fock space representation of $U_q(\hat{sl}_n)$ through natural ‘shift’ operators b_i .

We provide now a new approach to this q -fermionic Fock space via the theory of braided groups[6] as developed extensively by the author in recent years. We refer to [5] for a more recent review. The standard finite-dimensional quantum planes have such a braided group structure or

¹Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge. On leave 1995 + 1996 at the Department of Mathematics, Harvard University, Cambridge MA02138, USA

coaddition, which allows one to define braided differentiation[7], integration, epsilon tensors[8], differential forms, etc. on such spaces in a systematic way. Using such techniques, we explicitly derive the Heisenberg algebra action of [4] for the lowest non-trivial generators b_1, b_2 . Even these cases will be hard enough, but we believe that they demonstrate the possibility of a new approach using such techniques. Ultimately it may be possible to compute q -correlation functions themselves by such methods, which is one of the motivations for the work.

Our starting point is the infinite-dimensional quantum planes or exchange algebras, associated to unitary solutions of the parametrized Yang-Baxter equations

$$R_{12}(\frac{z}{w})R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(\frac{z}{w}), \quad R(z) = R(z^{-1})_{21}^{-1} \quad (1)$$

in a compact notation. Associated to this is the corresponding fermionic quantum plane $\Lambda(R(z))$ with

$$\theta_1(z)\theta_2(w) = -\theta_2(w)\theta_1(z)R(\frac{z}{w}), \quad \text{i.e.} \quad \theta_i(z)\theta_j(w) = \theta_b(w)\theta_a(z)R^a{}_i{}^b{}_j(\frac{z}{w}) \quad (2)$$

where $R(z) \in M_n \otimes M_n$ and $\theta(z)_i$, $i = 1, \dots, n$. There are also similar formulae without the - signs, for bosonic-type exchange algebras. The fermionic Fock space in [4] is of this general type (2), where, more precisely, the authors considered vector near to a chosen ‘vacuum vector’, rather than the algebra itself. We refer to [4] for details on this final step.

In Section 2, we study the algebra (2) for the entire class of solutions of (1) of the form

$$R(z) = \frac{R - zR_{21}^{-1}}{q - zq^{-1}}. \quad (3)$$

This Baxterisation formula solves (1) for *any* matrix solution R of the ordinary Yang-Baxter equations which is of Hecke type, in the sense

$$(PR - q)(PR + q^{-1}) = 0, \quad (4)$$

where P is the permutation matrix, which is the generality at which we work. This approach includes the $U_q(\hat{\mathfrak{sl}}_n)$ R-matrix as well as other more nonstandard systems. We show that the algebra $\Lambda(R(z))$ is an infinite ‘tensor product’ of copies of the fermionic quantum plane $\Lambda(R)$ with

$$\theta_1\theta_2 = -q\theta_2\theta_1R. \quad (5)$$

Such fermionic quantum planes have key properties from the theory of braided geometry, which we shall use. Among them is the braided coaddition

$$\Delta\theta = \theta \otimes 1 + 1 \otimes \theta, \quad (1 \otimes \theta_1)(\theta_2 \otimes 1) = -q^{-1}(\theta_2 \otimes 1)(1 \otimes \theta_1)R \quad (6)$$

where the two copies of $\Lambda(R)$ in $\Lambda(R) \underline{\otimes} \Lambda(R)$ enjoy the braid statistics shown (generalising the usual Bose-Fermi statistics of usual exterior algebras), which makes them braided groups rather than quantum groups. Moreover, because braided geometry works as well for fermionic as for bosonic spaces, its principal notions such as braided-differentiation, etc., work as well for $\Lambda(R)$ as for the more usual bosonic quantum planes. In particular, as a case of [7], we have braided differentiation on fermionic quantum planes[8]

$$\begin{aligned} \partial^i(\theta_1\theta_2\cdots\theta_m) &= e_1^i\theta_2\cdots\theta_m[m, -q^{-1}R]_{1\dots m} \\ \theta_1\theta_2\cdots\theta_m\overleftarrow{\partial^i} &= \theta_1\cdots\theta_{m-1}e_m^i\overline{[m; -q^{-1}R]}_{1\dots m} \\ [m, R]_{1\dots m} &= 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12}\cdots(PR)_{m-1m} \\ \overline{[m; R]}_{1\dots m} &= 1 + (PR)_{m-1m} + (PR)_{m-1m}(PR)_{m-2m-1} + \cdots + (PR)_{m-1m}\cdots(PR)_{12} \end{aligned} \quad (7)$$

as operators $\partial^i, \overleftarrow{\partial^i} : \Lambda(R) \rightarrow \Lambda(R)$. Here $(e^i)_j = \delta^i_j$ is a basis vector. One can also apply such ideas at the infinite-dimensional level (2), as functional differentiation, though we do not do so here.

Our goal is to make use of some of the rich structure of finite-dimensional braided spaces to study the infinite-dimensional fermionic Fock space. In effect, we study these exchange algebras as ‘braided wave functions’ where at each point (in momentum space) we have a mode θ^i behaving as a fermionic quantum plane. Moreover, our deriviations in this paper do not depend at any point on the precise form of the Hecke R-matrix. Hence we include not only the $U_q(\hat{sl}_n)$ theory but, in principle, generalise it to other non-standard affine quantum groups associated to the Baxterisation (3) of other Hecke R-matrices as well. We derive the Heisenberg algebra action in Section 3 in this setting. In Section 4, we conclude with some comments about covariance.

Some notations in the paper are as follows. Apart from the *braided integer matrices*[7] $[m, R]$ and $\overline{[m, R]}$ in (7), we also set

$$[m; q^{-2}] \equiv \frac{1 - q^{-2m}}{1 - q^{-2}}, \quad [m, n; R] \equiv (PR)_{mm+1}(PR)_{m+1m+2}\cdots(PR)_{n-1n}$$

$$\overline{[m, n; R]} \equiv (PR)_{n-1n} \cdots (PR)_{m+1m+2} (PR)_{mm+1}.$$

There is a change in conventions $q \rightarrow q^{-1}$ in our paper relative to [4]. Also, we write the fermionic quantum plane relations such as (5) in the even more compact form in which we suppress the numerical suffices entirely. Thus

$$\theta\theta \equiv \theta_1\theta_2, \quad \text{i.e.,} \quad \theta\theta = -q\theta\theta PR$$

is (5) in our notation: the tensor product of the vector indices θ_i is to be understood. When we do write numerical suffices θ_1, θ_2 etc, we henceforth mean the actual components of the vector θ . Finally, we write

$$\{\theta, \psi\}_R \equiv \theta\psi + q^{-1}\psi\theta PR, \quad \text{i.e.} \quad \{\theta_i, \psi_j\}_R \equiv \theta_i\psi_j + q^{-1}\psi_b\theta_a R^a{}_i{}^b{}_j$$

and sometimes $\mathbf{R} \equiv -q^{-1}R$, as useful shorthand notations.

Acknowledgements

These results were obtained during a visit in June 1995 to R.I.M.S. in Kyoto under a joint programme with the Isaac Newton Institute in Cambridge and the J.S.P.S. I would like to thank my host T. Miwa for extensive discussions.

2 Fermionic Fock space

The level 1 Fock space representation of $U_q(\hat{sl}_n)$ has been constructed in [1][2] and studied further in several papers, notably [3][4]. Here we take a slightly different point of view on this representation, taking as starting point the fermionic ‘exchange algebra’ $\Lambda(R(z))$ defined in (2). Our goal in this section is to break down the structure of this exchange algebra into many copies of standard finite-dimensional fermionic quantum planes $\Lambda(R)$ as in (5). We write $\theta(z) = \sum_{z \in \mathbb{Z}} \theta^i z^i$.

Theorem 2.1 *When $R(z)$ is of the form (3) (as in the sl_n case) then $\Lambda(R(z))$ is an infinite number of copies $\{\theta^i\}$ of the fermionic quantum plane $\Lambda(R)$ associated to the finite-dimensional R -matrix R , with relations*

$$\theta^i \theta^i (PR + q^{-1}) = 0, \quad \{\theta^i, \theta^{i-1}\}_R = 0$$

$$\{\theta^i, \theta^j\}_R = (q^{-2} - 1) \left(\sum_{s=1}^{s < \frac{i-j}{2}} \theta^{j+s} \theta^{i-s} (1 + q^{-2})^{s-1} (1 + PR) + \theta^{\frac{i+j}{2}} \theta^{\frac{i-j}{2}} q^{-2(\frac{i-j}{2}-1)} \right)$$

for $i - j > 1$. Here the last term is included only if $i - j$ is even.

Proof From the form of $R(z)$ we have

$$\sum_{i,j} (q - q^{-1} \frac{z}{w}) \theta^i \theta^j z^i w^j = \sum_{i,j} \theta^j w^j \theta^i z^i (PR - \frac{z}{w} (PR)^{-1}).$$

We equate powers of z, w , and hence require

$$\theta^j \theta^i PR + \theta^i \theta^j q = \theta^{j+1} \theta^{i-1} (PR)^{-1} + \theta^{i-1} \theta^{j+1} q^{-1}. \quad (8)$$

Considering the same equation with $i \rightarrow j+1$ and $j \rightarrow i-1$ and combining with (8) times qPR , gives

$$(\theta^i \theta^j + \theta^j \theta^i)(PR + q^{-1}) = 0, \quad (9)$$

on using the Hecke condition (4). This implies, in particular, that the θ^i modes each obey the finite-dimensional fermionic quantum plane algebra. Next, we consider (8) with $j = i-1$, i.e.,

$$\theta^{i-1} \theta^i PR + \theta^i \theta^{i-1} q = \theta^i \theta^{i-1} (PR)^{-1} + \theta^{i-1} \theta^i q^{-1}.$$

Combining with (9) and the Hecke condition $(PR)^2 = 1 + (q - q^{-1})PR$ gives $\{\theta^i, \theta^{i-1}\}_R = 0$ for neighbouring modes. Finally, for non-neighbouring modes, we use the Hecke condition to write (8) in the form

$$\{\theta^i, \theta^j\}_R = \{\theta^{i-1}, \theta^{j+1}\}_R + (q^{-2} - 1)(\theta^{i-1} \theta^{j+1} + \theta^{j+1} \theta^{i-1}), \quad (10)$$

which gives an inductive formula for $\{\theta^i, \theta^j\}_R$ in terms of ‘usual’ anticommutators of the intermediate modes. Alternatively, which we prefer, we use (9) and the Hecke condition to write (10) as

$$\{\theta^i, \theta^j\}_R = \{\theta^{i-1}, \theta^{j+1}\}_R (1 + (q - q^{-1})PR) + (q^{-2} - 1) \theta^{j+1} \theta^{i-1} (1 - q^{-1}PR). \quad (11)$$

Using this, we obtain the formula stated for the ordering relations between non-adjacent modes, by induction. Note that, by the Hecke condition (4), $(1 - q^{-1}PR)PR = (1 - q^{-1}PR)(-q^{-1})$. The start of the induction is when the i, j are equal or one apart (as $i - j$ is even or odd), which cases we have already computed separately. We see that between adjacent modes there are the usual braid statistics associated to two copies of the finite-dimensional fermionic quantum plane

(as needed for their braided coaddition structure in (6)). Between modes that are further apart, we have the same ‘leading’ braid statistics + decendent terms involving intermediate modes. \square

The algebra in this theorem is computed formally from the powerseries, but can afterwards be taken as a definition of the exchange algebra, as generated by θ^i . We proceed now on this basis. We see that each of the modes has a geometrical picture as the algebra $\Lambda(R)$ of q -differential forms; see [8] for the braided-geometrical construction (starting from the braided coaddition law). In particular, in nice cases (such as the sl_n case), each has a top form

$$\omega^i = \theta_1^i \cdots \theta_n^i$$

with all others of this degree being multiples of it. The products $\theta^i \omega^i$ are zero for all i . There is also an underlying bosonic space with $\theta^i = d\mathbf{x}^i$, where \mathbf{x}^i obey $\mathbf{x}^i \mathbf{x}^i = \mathbf{x}^i \mathbf{x}^i q^{-1} PR$. We do not use this full geometrical picture here, regarding the θ^i as intrinsic fermionic-type coordinates in their own right.

It is worth noting that our fermionic Fock space algebra in Theorem 2.1 is clearly a more complicated variant of the actual braided tensor product algebra $\underline{\otimes}_{i=-\infty}^{\infty} \Lambda^i(R)$ with relations

$$\theta^i \theta^i (PR + q^{-1}) = 0, \quad \{\theta^i, \theta^j\}_R = 0 \quad (12)$$

for all $i > j$. This algebra was discussed in [6], where it was proposed as a discrete model of the exchange algebra in 2-D quantum gravity[9]. Indeed, one can consider it as a fermionic exchange algebra for the discretely (and additively) parametrised R-matrix

$$R(i-j) = \begin{cases} q^{-1}R & i > j \\ qR & i = j \\ qR_{21}^{-1} & i < j. \end{cases} \quad (13)$$

The algebra (12), although pertaining to a different model than the one above (and with i as a discrete version of a position variable rather than a mode label), nevertheless has a similar form to our fermionic Fock space in Theorem 2.1, just without the descendent modes. Moreover, its construction as a braided tensor product (with relations as in (6) between different modes) ensures that it remains covariant under (a dilatonic extension of) $U_q(sl_n)$ or other quantum group (according to the R-matrix). By contrast, the more complicated fermionic Fock space in Theorem 2.1 is covariant under $U_q(\hat{sl}_n)$ or other affine quantum group.

3 Computation of the Heisenberg algebra action

It is clear from the form of the relations (8) of $\Lambda(R(z))$ that

$$b_i : \Lambda(R(z)) \rightarrow \Lambda(R(z)), \quad b_i(\theta^j) = \theta^{j+i} \quad (14)$$

is a derivation on the algebra, for each i . It is shown in [4], (by Hecke algebra and vertex operator methods) that these b_i define an action of the Heisenberg algebra according to

$$[b_i, b_{-j}] = \delta_{i,j} i \left(\frac{1 - q^{-2ni}}{1 - q^{-2i}} \right), \quad (15)$$

when acting on

$$\omega = \omega^0 \omega^1 \dots$$

or vectors near to this (differing only in finitely many coefficients). We show now how this result can alternatively be obtained by braided-geometrical methods. Note that ω is in a completion of the algebra generated by the modes. However, all our operations stay within the space of vectors near to it, and hence remain algebraic; see [4] for a more formal way to say this.

Proposition 3.1 *For $i \geq 1$, we have*

$$b_i(\omega) = 0, \quad b_{-i}(\omega) = b_{-i}(\omega^0) \omega^1 \dots + \omega^0 b_{-i}(\omega^1) \omega^2 \dots + \dots + \omega^0 \omega^1 \dots \omega^{i-2} b_{-i}(\omega^{i-1}) \omega^i \dots.$$

Proof Firstly, $b_i(\omega) = 0$ for $i \geq 1$ since $b_i(\omega^j)$ has in it modes θ^{j+i} ; moving these to the right using the braided-anticommutation relations with $\theta^j, \theta^{j+1}, \dots, \theta^{j+i-1}$, gives eventually $\theta^{j+i} \omega^{j+i} = 0$. Along the way, if $i \geq 2$, we generate descendents which lie in the range $\theta^{j+1}, \dots, \theta^{j+i-1}$; moving each of these to the right kills these as well. Similarly for their descendents, etc.

For b_{-i} we have

$$b_{-i}(\omega^j) = \theta_1^{j-i} \theta_2^j \dots \theta_n^j + \dots + \theta_1^j \dots \theta_{n-1}^j \theta_n^{j-i} = \theta_{a_1}^{j-i} \theta_{a_2}^j \dots \theta_{a_n}^j [n; \mathbf{R}]_{1 \dots n}^{a_1 \dots a_n} + \text{descendents}$$

where the descendents involve $\theta^{j-i+1}, \dots, \theta^{j-1}$. We moved θ^{j-2} to the left in each term, just as in the definition of braided differentiation[7], but now picking up descendents from the right hand side of the anticommutators in Theorem 2.1.

Hence, when we compute $b_{-i}(\omega)$ as a derivation, only the first i terms contribute, as stated; the $\omega^0 \dots \omega^{j-1} b_{-i}(\omega^j)$ for $j \geq i$ do not contribute because the terms of $b_{-i}(\omega^j)$ each contain a

mode in the range $\theta^{j-i}, \dots, \theta^{j-1}$ which, using the relations in Theorem 2.1, can be pushed left until it multiplies one of $\omega^{j-i}, \dots, \omega^{j-1}$, and thereby vanishes. The descendents generated in this process when $i \geq 2$ can likewise be pushed to the left and annihilated. Similarly for their descendents, etc. \square

The simplest case of (15) follows trivially:

Proposition 3.2 $b_{-1}(\omega^j) = \theta_{a_1}^{j-1} \theta_{a_2}^j \dots \theta_{a_n}^j [n; \mathbf{R}]_{1\dots n}^{a_1 \dots a_n}$. Hence $[b_1, b_{-1}] = [n, q^{-2}]$ when acting on ω .

Proof In this case θ^{j-1} is adjacent to θ^j so no descendents are generated when we move it to the left in each term of $b_{-1}(\omega^j)$. Hence $b_{-1}(\omega) = \theta^{-1} \theta^0 \dots \theta^0 [n; \mathbf{R}] \omega^1 \dots$. When we apply b_1 to this, only the action on θ^{-1} contributes: other modes have degree ≥ 1 and annihilate when moved to the right. Hence $b_1(b_{-1}(\omega)) = \theta^0 \dots \theta^0 [n; \mathbf{R}] \omega^1 \dots$. On the other hand, PR acts as $-q^{-1}$ on $\theta\theta$ (the defining relations of each mode $\Lambda(R)$ in Theorem 2.1). Hence $[n; \mathbf{R}]$ can be replaced by $[n; q^{-2}]$ when acting on $\Lambda^{(0)}(R)$. \square

The same techniques apply for the action of the higher Heisenberg generators. We do the computation now for $[b_2, b_{-2}]$.

Lemma 3.3

$$\begin{aligned} b_{-2}(\omega^j) &= \theta^{j-2} \theta^j \dots \theta^j [n; \mathbf{R}]_{1\dots n} \\ &\quad + (q^{-2} - 1) \theta^{j-1} \theta^{j-1} \theta^j \dots \theta^j ([n-1; \mathbf{R}]_{2\dots n} + [2, 3; \mathbf{R}][1, 2; \mathbf{R}][n-2; \mathbf{R}]_{3\dots n} \\ &\quad + \dots + [2, n-1; \mathbf{R}][1, n-2; \mathbf{R}][2; \mathbf{R}]_{n-1n} + [2, n; \mathbf{R}][1, n-1; \mathbf{R}]). \end{aligned}$$

Hence

$$b_2(b_{-2}(\omega^0))\omega^1 \dots = \left([n; q^{-2}] + (1 - q^{-2}) \left([n-1; q^{-4}] - q^{-2(n-1)} [n-1; q^{-2}] \right) \right) \omega.$$

Proof Clearly,

$$\begin{aligned} b_{-2}(\omega^j) &= \theta_1^{j-2} \theta_2^j \dots \theta_n^j + \dots + \theta_1^j \dots \theta_{n-1}^j \theta_n^{j-2} \\ &= \theta^{j-2} \theta^j \dots \theta^j [n, \mathbf{R}]_{1\dots n} + (q^{-2} - 1) \theta_1^{j-1} \theta^{j-1} \theta^j \dots \theta^j [n-1; \mathbf{R}]_{2\dots n} \\ &\quad + (q^{-2} - 1) \theta_1^j \theta_2^{j-1} \theta^{j-1} \theta^j \dots \theta^j [n-2; \mathbf{R}]_{3\dots n} + \dots + (q^{-2} - 1) \theta_1^j \dots \theta_{n-2}^j \theta_{n-1}^{j-1} \theta_n^{j-1}, \end{aligned}$$

where we use

$$\theta^j \theta^{j-2} = \theta^{j-2} \theta^j P\mathbf{R} + (q^{-2} - 1) \theta^{j-1} \theta^{j-1}$$

from Theorem 2.1. We move each θ^{j-2} to the left at the price of a factor $P\mathbf{R}$ and a $\theta^{j-1} \theta^{j-1}$.

We then add up all the descendents as generated in each position.

From this expression, the expression stated for $b_{-2}(\omega^j)$ follows at once: in each of the descendent terms, we move $\theta^{j-1} \theta^{j-1}$ to the left, accumulating powers of $P\mathbf{R}$ for each one.

Then $b_2(b_{-2}(\omega^0))\omega^1 \dots$ is computed as follows. When we apply b_2 , only its action on the θ^{-2} mode or the first θ^{-1} mode in $b_{-2}(\omega^0)$ can contribute, since the other cases produce modes which can be pushed to the right and annihilated, along with their descendents. The first of these gives $\theta^0 \dots \theta^0[n; \mathbf{R}]\omega^1 \dots = \omega[n; q^{-2}]$ by the relations in $\Lambda^{(0)}(R)$. The second case contains $\theta^1 \theta^{-1} \theta^0 \theta^0 \dots \theta^0$ where θ^1 can also be pushed to the right and annihilated. In the process, however, it contributes a descendent

$$\theta^0 \theta^0 \dots \theta^0 (q^{-2} - 1)^2 ([n - 1; \mathbf{R}]_{2 \dots n} + [2, 3; \mathbf{R}][1, 2; \mathbf{R}][n - 2; \mathbf{R}]_{3 \dots n} + \dots + [2, n; \mathbf{R}][1, n - 1; \mathbf{R}]) \omega^1 \dots.$$

Finally, using the relations in $\Lambda^{(0)}(R)$, we can replace $P\mathbf{R}$ by q^{-2} , giving

$$\begin{aligned} (q^{-2} - 1)^2 & \left([n - 1; q^{-2}] + q^{-4}[n - 2; q^{-2}] + \dots q^{-4(n-2)}[1; q^{-2}] \right) \omega \\ & = (1 - q^{-2}) \left([n - 1; q^{-4}] - q^{-2(n-1)}[n - 1; q^{-2}] \right) \omega \end{aligned}$$

as stated. \square

By a strictly analogous computation, we have

$$\begin{aligned} b_2(\omega^j) &= \theta^j \dots \theta^j \theta^{j+2} \overline{[n; \mathbf{R}]_{1 \dots n}} \\ &+ (q^{-2} - 1) \theta^j \dots \theta^j \theta^{j+1} \theta^{j+1} (\overline{[n - 1; \mathbf{R}]_{1 \dots n-1}} + \overline{[1, 2; \mathbf{R}][2, 3; \mathbf{R}][n - 2; \mathbf{R}]_{1 \dots n-2}} \\ &+ \dots + \overline{[1, n - 2; \mathbf{R}][2, n - 1; \mathbf{R}][2; \mathbf{R}]_{12}} + \overline{[1, n - 1; \mathbf{R}][2, n; \mathbf{R}]}, \end{aligned}$$

showing its descendents explicitly. Here we moved θ^{j+2} to the right, and the resulting descendents also to the right.

Proposition 3.4 $[b_2, b_{-2}] = 2 \left(\frac{1 - q^{-4n}}{1 - q^{-4}} \right)$ when acting on ω .

Proof We are now ready to compute

$$b_2(b_{-2}(\omega)) = b_2(b_{-2}(\omega^0)\omega^1 + \omega^0 b_{-2}(\omega^1))\omega^2 \dots$$

where $b_2(\omega^2)\omega^3$ etc., do not contribute, as in Proposition 3.1 (shifted down by translation invariance). The first term is the same as $b_2(b_{-2}(\omega^0))\omega^1 \dots$ (for the same reason) and was computed in Lemma 3.3. The second term is

$$\begin{aligned} b_2(\omega^0 b_{-2}(\omega^1))\omega^2 \dots &= b_2(\theta^0 \dots \theta^0 \theta_{a_1}^{-1} \theta_{a_2}^1 \dots \theta_{a_n}^1 [n; \mathbf{R}]_{1 \dots n}^{a_1 \dots a_n})\omega^2 \dots \\ &= b_2(\theta^{-1} \theta^0 \dots \theta^0 [1, n+1; \mathbf{R}]_{1 \dots na_1} \theta_{a_2}^1 \dots \theta_{a_n}^1 [n; \mathbf{R}]_{1 \dots n}^{a_1 \dots a_n})\omega^2 \dots \\ &= \theta^1 \theta^0 \dots \theta^0 [1, n+1; \mathbf{R}]_{1 \dots na_1} \theta_{a_2}^1 \dots \theta_{a_n}^1 [n; \mathbf{R}]_{1 \dots n}^{a_1 \dots a_n} \omega^2 \dots \\ &= \theta^0 \dots \theta^0 \theta^1 \overline{[1, n+1; \mathbf{R}]} [1, n+1; \mathbf{R}]_{1 \dots na_1} \theta_{a_2}^1 \dots \theta_{a_n}^1 [n; \mathbf{R}]_{1 \dots n}^{a_1 \dots a_n} \omega^2 \dots \end{aligned}$$

where the descendents in $b_{-2}(\omega^1)$ annihilate against ω^0 to the left, and so do not contribute in the first line. We move the θ^{-1} mode to the left in the second line, picking up powers of $P\mathbf{R}$. The third equality then applies b_2 . Only its action on θ^{-1} contributes, since modes θ^2 or higher can be moved to the right and annihilate. The fourth equality moves the resulting θ^1 to the right, picking up powers of $P\mathbf{R}$ again.

We now use the Hecke condition in the form $(P\mathbf{R})^2 = q^{-2} + (q^{-2} - 1)P\mathbf{R}$ and the Yang-Baxter equations in the form $(P\mathbf{R})_{23}(P\mathbf{R})_{12}(P\mathbf{R})_{23} = (P\mathbf{R})_{12}(P\mathbf{R})_{23}(P\mathbf{R})_{12}$ repeatedly, to observe that

$$\begin{aligned} &\theta^0 \dots \theta^0 \theta^1 \overline{[1, n+1; \mathbf{R}]} [1, n+1; \mathbf{R}] \\ &= \theta^0 \dots \theta^0 \theta^1 (q^{-2} \overline{[2, n+1; \mathbf{R}]} [2, n+1; \mathbf{R}] \\ &\quad + (q^{-2} - 1)(P\mathbf{R})_{nn+1} \dots (P\mathbf{R})_{23}(P\mathbf{R})_{12}(P\mathbf{R})_{23} \dots (P\mathbf{R})_{nn+1}) \\ &= \theta^0 \dots \theta^0 \theta^1 \left(q^{-2} \overline{[2, n+1; \mathbf{R}]} [2, n+1; \mathbf{R}] + (q^{-2} - 1)[1, n; \mathbf{R}](P\mathbf{R})_{nn+1} \overline{[1, n; \mathbf{R}]} \right) \\ &= \theta^0 \dots \theta^0 \theta^1 \left(q^{-2} \overline{[2, n+1; \mathbf{R}]} [2, n+1; \mathbf{R}] + (q^{-2} - 1)q^{-2(n-1)} \overline{[1, n+1; \mathbf{R}]} \right) \\ &= \dots = \theta^0 \dots \theta^0 \theta^1 \left(q^{-2n} + (q^{-2} - 1)q^{-2(n-1)} (\overline{[n+1; \mathbf{R}]} - 1) \right) \\ &= \theta^0 \dots \theta^0 \theta^1 \left(q^{-2n} - (q^{-2} - 1)q^{-2(n-1)} \right) = \theta^0 \dots \theta^0 \theta^1 q^{-2(n-1)}. \end{aligned}$$

The third equality replaces $P\mathbf{R}$ by q^{-2} in $[1, n; \mathbf{R}]$ since it acts on $\theta^0 \dots \theta^0$ to its left. We then iterate these steps, collecting the $\overline{[n+1; \mathbf{R}]}$ which are generated in this way as $\overline{[n+1; \mathbf{R}]} - 1$.

Finally, we note that

$$\theta^0 \dots \theta^0 \theta^1 \overline{[n+1; \mathbf{R}]} = \theta^0 \dots \theta^0 \overleftarrow{\partial} \cdot \theta^1 = 0$$

since on the right hand side we have the braided differential of $n+1$ copies of θ^0 , which vanishes.

With this result, we can complete our calculation as

$$b_2(\omega^0 b_{-2}(\omega^1)) \omega^2 \dots = \omega^0 q^{-2(n-1)} \theta_{a_1}^1 \dots \theta_{a_n}^1 [n; \mathbf{R}]_{1 \dots n}^{a_1 \dots a_n} \omega^2 \dots = q^{-2(n-1)} [n; q^{-2}] \omega$$

since $P\mathbf{R}$ can be replaced by q^{-2} when acting on the algebra $\Lambda^{(1)}(R)$.

Adding this contribution to that from Lemma 3.3, we find

$$b_2(b_{-2}(\omega)) = \left([n; q^{-2}] (1 + q^{-2(n-1)}) + (1 - q^{-2}) ([n-1; q^{-4}] - q^{-2(n-1)} [n-1; q^{-2}]) \right) \omega$$

which computes to the final result stated. \square

Although we have only covered the $i = 1, 2$ cases of (15) in this paper, it is clear that the method introduced here can provide a viable alternative to the vertex operator proof in [4]. Since the approach there uses directly the correlation function for XXZ vertex operators, our direct ‘braided geometric’ technique implies in principle a new approach to the computation of these.

4 Concluding remarks

It is significant that all computations in this paper have been made without reference to any specific details of the R -matrix, so long as it is Hecke type. This means that the fermionic Fock space construction in [4] works quite generally; it may be interesting to consider some non-standard examples. A further question is how to extend the above methods to non-Hecke cases such as the affine quantum group $U_q(\hat{so}_3)$. Related to this is the construction of higher level fermionic Fock space representations, even for $U_q(\hat{sl}_2)$. For these one should make semi-infinite tensor products of fermionic quantum planes where the underlying finite-dimensional R -matrix is not of Hecke type. We note that the Baxterisation formula for the parametrised R -matrix in the \hat{so}_n case is indeed known, though having now a more complicated form. Hence in principle our ‘decomposition’ methods might be applied.

Also, in braided geometry the fermionic quantum planes (like other quantum planes) are fully covariant not exactly under $U_q(\hat{sl}_n)$ (or other quantum group, according to the R -matrix)

but under a dilatonic extension of it. This is needed whenever the quantum plane normalisation is not the quantum group normalisation of the R-matrix. Analogously, the fermionic Fock space is not quite covariant under the quantum loop group associated to $R(z)$ but under its central extension, which in our case is $U_q(\hat{sl}_n)$. Formally, and before considering the R-matrix normalisation, the exchange algebra (2) should be covariant under the quantum loop group in the R-matrix form with generators $I^\pm(z)$, which would make it a level 0 module of $U_q(\hat{sl}_n)$. Hence it appears that similar ‘dilaton’ effects are responsible for the anomaly which makes the fermionic Fock space considered above into level 1. This is another direction for further work.

References

- [1] T. Hayashi *Commun. Math. Phys.*, 127:129–144, 1990.
- [2] K. Misra and T. Miwa. Crystal base for the basic representation of $U_q(\hat{sl}_n)$. *Commun. Math. Phys.* 134: 79–88, 1990.
- [3] E. Stern *Internat. Math. Res. Notices*, 4:201–220, 1995.
- [4] M. Kashiwara, T. Miwa and E. Stern. Decomposition of q -deformed Fock space. R.I.M.S. preprint, 1995.
- [5] S. Majid. Introduction to braided geometry and q -Minkowski space. Proc. School ‘Enrico Fermi’ CXXVII, Varenna. IOS Press, 1995.
- [6] S. Majid. Beyond supersymmetry and quantum symmetry (an introduction to braided groups and braided matrices). In H.J. de Vega and M.-L. Ge, eds., *Quantum Groups, Integrable Statistical Models and Knot Theory*, pp. 231–282. World Sci., 1993.
- [7] S. Majid. Free braided differential calculus, braided binomial theorem and the braided exponential map. *J. Math. Phys.*, 34:4843–4856, 1993.
- [8] S. Majid Epsilon tensor for quantum and braided spaces. *J. Math. Phys.* 36:1991–2007, 1995.
- [9] J.-L. Gervais. The quantum group structure of 2d gravity and minimal models, I. *Commun. Math. Phys.* 130:257, 1990.